MINDLAB

IBDP Mathematics AA Revision Notes

Topic 5.8: Optimization — Maximum and Minimum Problems

Syllabus Reference

SL 5.8 – Testing for maximum or minimum. Optimization problems.

Learning Objectives

Upon completion of this section, students should be able to:

- Set up optimization problems from word descriptions
- Express quantities to be optimized as functions of one variable
- Use constraints to eliminate variables
- Apply calculus to find maximum and minimum values
- Verify solutions and interpret results in context

1 Introduction to Optimization

Optimization

Optimization is the process of finding the maximum or minimum value of a quantity subject to given constraints.
Common applications include:

- Maximizing profit, area, volume, efficiency
- Minimizing cost, material usage, distance, time

Types of Extrema

- **Local (relative) maximum/minimum:** Largest/smallest value in a neighborhood
- **Global (absolute) maximum/minimum:** Largest/smallest value over the entire domain

For optimization problems, we usually want the **global** extremum within the given constraints.

2 General Method for Optimization

Steps for Solving Optimization Problems

Step 1: Understand the problem

- Read carefully and identify what needs to be maximized/minimized
- Draw a diagram if applicable
- Define variables clearly

Step 2: Write the objective function

- Express the quantity to optimize as a function
- Initially, this may involve multiple variables

Step 3: Use constraints

- Identify any relationships or restrictions given
- Use these to express the objective function in terms of ONE variable

Step 4: Find critical points

- Differentiate the objective function
- Solve $\frac{dy}{dx} = 0$ (or $\frac{dA}{dx} = 0$, etc.)

Step 5: Verify it's a maximum or minimum

- Use second derivative test: $f''(x) < 0$ (max), $f''(x) > 0$ (min)
- OR check endpoints if domain is restricted
- OR use first derivative sign change

Step 6: Answer the question

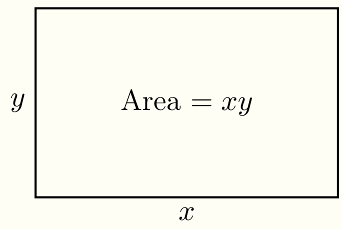
- Calculate the required values
- Give answer in context with appropriate units

3 Optimization with Geometric Constraints

Maximizing Area with Fixed Perimeter

A farmer has 100 meters of fencing to enclose a rectangular field. Find the dimensions that maximize the enclosed area.

Solution:



Step 1: Define variables
Let x = length, y = width of the rectangle.

Step 2: Objective function
We want to maximize: $A = xy$

Step 3: Constraint
Perimeter: $2x + 2y = 100$
Solve for y: $y = 50 - x$
Substitute into area: $A = x(50 - x) = 50x - x^2$

Step 4: Differentiate and find critical points

$$\frac{dA}{dx} = 50 - 2x = 0$$

$$x = 25$$

Step 5: Verify maximum

$$\frac{d^2A}{dx^2} = -2 < 0 \implies \text{Maximum}$$

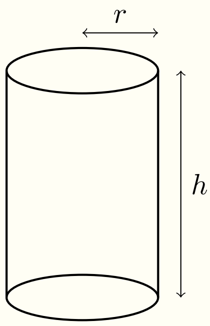
Step 6: Find dimensions
$x = 25$ m, $y = 50 - 25 = 25$ m
Maximum area $= 25 \times 25 = 625$ m^2

Answer: The field should be a square with side 25 m for maximum area of 625 m^2.

Minimizing Surface Area

A closed cylindrical can must have a volume of 500 cm^3. Find the dimensions that minimize the surface area.

Solution:



Step 1: Define variables
Let r = radius, h = height
Step 2: Objective function
Surface area: $S = 2\pi r^2 + 2\pi rh$ (two circles + curved surface)
Step 3: Constraint
Volume: $V = \pi r^2 h = 500$
Solve for h: $h = \frac{500}{\pi r^2}$
Substitute:

$$S = 2\pi r^2 + 2\pi r \cdot \frac{500}{\pi r^2} = 2\pi r^2 + \frac{1000}{r}$$

Step 4: Differentiate

$$\frac{dS}{dr} = 4\pi r - \frac{1000}{r^2}$$

Set equal to zero:

$$4\pi r = \frac{1000}{r^2}$$

$$4\pi r^3 = 1000$$

$$r^3 = \frac{250}{\pi}$$

$$r = \sqrt[3]{\frac{250}{\pi}} \approx 4.30 \text{ cm}$$

Step 5: Verify minimum

$$\frac{d^2S}{dr^2} = 4\pi + \frac{2000}{r^3}$$

Since $r > 0$, we have $\frac{d^2S}{dr^2} > 0$, confirming a minimum.
Step 6: Find height

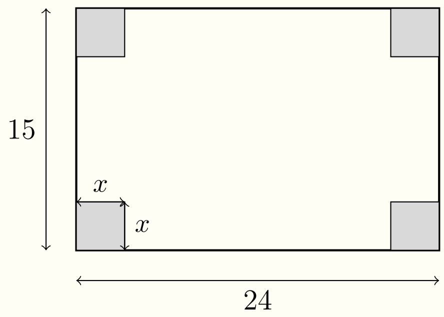
$$h = \frac{500}{\pi r^2} = \frac{500}{\pi \cdot (250/\pi)^{2/3}} = \frac{500}{\pi^{1/3} \cdot 250^{2/3}}$$

After simplification: $h = 2r = 2\sqrt[3]{\frac{250}{\pi}} \approx 8.60$ cm
Answer: Radius $\approx$ 4.30 cm, Height $\approx$ 8.60 cm
Note: The optimal cylinder has $h = 2r$ (height equals diameter).

Box with Maximum Volume

A rectangular sheet of cardboard measuring 24 cm by 15 cm has squares of side x cut from each corner. The sides are then folded up to form an open box. Find the value of x that maximizes the volume.

Solution:



Step 1: Express dimensions of box
After folding:

- Length: $24 - 2x$
- Width: $15 - 2x$
- Height: x

Step 2: Volume function

$$V = x(24 - 2x)(15 - 2x)$$

Expand:

$$V = x(360 - 48x - 30x + 4x^2) = x(4x^2 - 78x + 360)$$

$$V = 4x^3 - 78x^2 + 360x$$

Step 3: Constraints
For valid box: $x > 0$, $24 - 2x > 0$, $15 - 2x > 0$
So: $0 < x < 7.5$
Step 4: Differentiate

$$\frac{dV}{dx} = 12x^2 - 156x + 360 = 12(x^2 - 13x + 30)$$

Set equal to zero:

$$x^2 - 13x + 30 = 0$$

$$(x - 3)(x - 10) = 0$$

$$x = 3 \text{ or } x = 10$$

Since $0 < x < 7.5$, we have $x = 3$ cm.
Step 5: Verify maximum

$$\frac{d^2V}{dx^2} = 24x - 156$$

At $x = 3$: $\frac{d^2V}{dx^2} = 72 - 156 = -84 < 0 \implies$ Maximum
Step 6: Calculate maximum volume

$$V = 3(24 - 6)(15 - 6) = 3 \times 18 \times 9 = 486 \text{ cm}^3$$

Answer: $x = 3$ cm gives maximum volume of 486 cm^3.

4 Optimization in Economics

Profit Maximization

A company's profit function is $P(x) = -2x^2 + 120x - 800$ dollars, where x is the number of units sold. Find the number of units that maximizes profit and the maximum profit.
Solution:
Step 1: The objective function is already given: $P(x) = -2x^2 + 120x - 800$
Step 2: Differentiate

$$\frac{dP}{dx} = -4x + 120$$

Step 3: Find critical point

$$-4x + 120 = 0 \implies x = 30$$

Step 4: Verify maximum

$$\frac{d^2P}{dx^2} = -4 < 0 \implies \text{Maximum}$$

Step 5: Calculate maximum profit

$$P(30) = -2(900) + 120(30) - 800 = -1800 + 3600 - 800 = 1000$$

Answer: Sell 30 units for maximum profit of \$1000.

Minimizing Cost

The cost of running a car at speed v km/h is $C = \frac{v^2}{100} + \frac{900}{v}$ dollars per hour. Find the speed that minimizes cost.
Solution:
Step 1: Differentiate

$$\frac{dC}{dv} = \frac{2v}{100} - \frac{900}{v^2} = \frac{v}{50} - \frac{900}{v^2}$$

Step 2: Set equal to zero

$$\frac{v}{50} = \frac{900}{v^2}$$

$$v^3 = 45000$$

$$v = \sqrt[3]{45000} \approx 35.6 \text{ km/h}$$

Step 3: Verify minimum

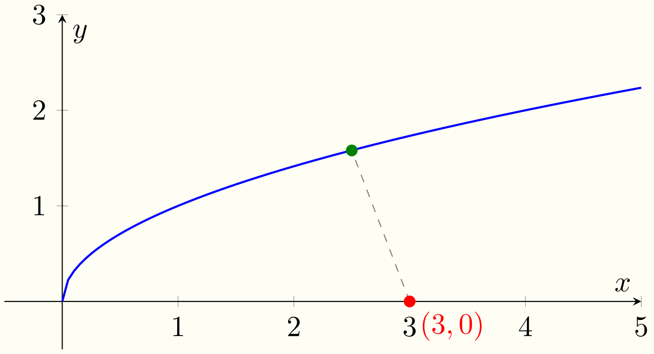
$$\frac{d^2C}{dv^2} = \frac{1}{50} + \frac{1800}{v^3}$$

For $v > 0$: $\frac{d^2C}{dv^2} > 0 \implies$ Minimum
Answer: Optimal speed is approximately 35.6 km/h.

5 Distance Problems

Minimum Distance to a Curve

Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$.
Solution:



Let the point on the curve be $(x, \sqrt{x})$.
Distance formula:

$$D = \sqrt{(x-3)^2 + (\sqrt{x}-0)^2}$$

Tip: Minimize D^2 instead (same result, easier calculus):

$$D^2 = (x-3)^2 + x = x^2 - 6x + 9 + x = x^2 - 5x + 9$$

Let $S = D^2$. Then:

$$\frac{dS}{dx} = 2x - 5 = 0 \implies x = 2.5$$

Verify: $\frac{d^2S}{dx^2} = 2 > 0 \implies$ Minimum
Find the point: $y = \sqrt{2.5} = \frac{\sqrt{10}}{2}$
Answer: The closest point is $\left(\frac{5}{2}, \frac{\sqrt{10}}{2}\right)$ or $(2.5, 1.58)$.

6 GDC Techniques for Optimization

Using GDC for Optimization

Method 1: Graph and find maximum/minimum

1. Enter the objective function in $Y1$
2. Graph with appropriate window
3. Use `2nd` → `CALC` → `maximum` or `minimum`
4. Set left bound, right bound, and guess

Method 2: Solve derivative = 0

1. Enter $f(x)$ in $Y1$
2. Enter $f'(x)$ in $Y2$ (use nDeriv or calculate manually)
3. Find zeros of $Y2$ using `2nd` → `CALC` → `zero`

Method 3: Table analysis

1. Set up TABLE with appropriate ΔTbl
2. Scroll to find where function reaches extreme value

Common Mistakes in Optimization

1. **Not checking if critical point is max or min:**
   Always verify using second derivative test or boundary analysis.
2. **Forgetting to check domain boundaries:**
   The optimal value might occur at an endpoint, not a critical point.
3. **Wrong variable elimination:**
   Be careful when using constraints to eliminate variables.
4. **Not answering the actual question:**
   If asked for "dimensions," give all dimensions, not just the variable x.
5. **Missing units:**
   Always include appropriate units in your final answer.
6. **Setting up wrong objective function:**
   Read carefully: maximize AREA vs maximize PERIMETER are different!

IB Exam Advice

- Draw a clear diagram and label all variables
- Write the constraint equation and objective function explicitly
- Show all differentiation steps
- Verify your answer makes sense in context (e.g., dimensions can't be negative)
- State clearly whether you've found a maximum or minimum and why
- On calculator papers, you can use GDC to verify, but show the calculus method
- Check your answer by substituting back into original expressions

7 Topic Links

- **Topic 5.6:** Differentiation rules needed for finding derivatives
- **Topic 5.7:** Second derivative test for classifying extrema
- **Topic 5.2:** Understanding increasing/decreasing functions
- **Topic 3:** Geometry formulas for area, volume, surface area
- **Topic 5.14 (AHL):** Related rates — dynamic optimization

8 Practice Problems

1. A rectangular enclosure is to be made using 200 m of fencing. One side is against a wall (no fence needed). Find the dimensions that maximize the area.
2. Find two positive numbers whose sum is 50 and whose product is maximum.
3. An open-top box is made from a square piece of cardboard (side 30 cm) by cutting equal squares from corners and folding up. Find the side of the cut square that maximizes volume.
4. A cylindrical can (open top) must hold 1000 cm^3. Find the radius that minimizes surface area.
5. The profit from selling x items is $P(x) = 250x - x^2 - 5000$. Find the number of items to maximize profit.
6. Find the point on the line $y = 2x + 1$ closest to the origin.

Answers:

1. 50 m × 100 m (100 m side against wall); Area = 5000 m^2
2. Both numbers are 25; Product = 625
3. 5 cm; Maximum volume = 2000 cm^3
4. $r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42$ cm

5. 125 items; Maximum profit = \$10,625
6. $\left(-\frac{2}{5}, \frac{1}{5}\right)$